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LETTER TO THE EDITOR

Semiclassical approach to ground states within the Klein–Gordon equation

N A Kobylinsky†§, S S Stepanov‡ and R S Tutik‡

† Institute for Theoretical Physics, Kiev, 252130, USSR

‡ Department of Physics, Dniepropetrovsk State University, Dniepropetrovsk, 320625, USSR

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Abstract. A new tool for deriving Regge trajectories and energy eigenvalues for ground states of central potentials is applied to the Klein-Gordon equation. Based upon the $\hbar$-expansion, the simple recursion formulae are presented. The problems of the π-mesonic atom and funnel-shaped potential are treated with this technique.

Several attempts have been made to treat the relativistic aspects of an arbitrary bound state in static potentials in the Klein-Gordon equation. For most potentials, however, this equation is not exactly soluable and one, therefore, has to resort to some approximation scheme. In particular, Au and Aharonov (1981), Lai (1982), and Rogers (1985) used a logarithmic perturbation theory to determine the energy eigenvalues $E_{n,l}$ with integer orbital (l) and radial (n) quantum numbers; at that time Miramontes and Pajares (1984) and Chatterjee (1986) employed the non-perturbative $1/N$-expansion method for this goal (N is the spatial dimensionality).

But in hadron physics it is more convenient to formulate the problem otherwise and to calculate $l_n(E)$, i.e. parent ($n=0$) and daughter ($n=1,2,\ldots$) Regge trajectories $\alpha(E)$ (Collins 1977). Since in a system with a discrete spectrum the Regge trajectories are just the inverse functions of the energy eigenvalues considered as analytic functions of the angular momentum variable, one can resort to usual quantum mechanical methods. Within the Klein-Gordon equation the useful, although quite bulky, perturbative technique for the Regge trajectory expansion has been developed by Müller (1965) with application to a Yukawa-type potential and spread to a general even power potential (Sharma and Iyer 1982). A method based on the use of continued fractions was also offered (Bhargava and Sharma 1983).

Recently, a powerful new tool for deriving Regge trajectories and energy eigenvalues for bound states of central potentials within the Schrödinger equation was proposed by Kobylinsky *et al* (1989). Based upon the semiclassical expansions in Planck's constant $\hbar$, this method is non-perturbative in nature and it can be calculated by a simple algebraic recursion method. In the present letter, we would like to extend this technique to a relativistic case in which the Klein-Gordon equation is used.

The radial part of the Klein-Gordon equation (in units $C=1$) for a scalar particle of mass m in the presence of a fourth-component Lorentz vector potential $V_v(r)$ and a scalar potential $V_s(r)$ is given by

$$\hbar^2 U''(r) = \left((m+V_s(r))^2 - (E - V_v(r))^2 + \frac{\hbar^2 l(l+1)}{r^2}\right) U(r) \tag{1}$$

§ Deceased 30 April 1989.

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where a prime denotes a derivative with respect to its argument. Expressed, for simplicity, in the form

$$\hbar^2 U''(r) = [G(r, E) + \Lambda/r^2]U(r) \tag{2}$$

with $\Lambda = \hbar^2 l(l+1)$ and potential function

$$G(r, E) = (m + V_s(r))^2 - (E - V_v(r))^2 \tag{3}$$

this equation is the subject of the following discussion.

For the sake of brevity we shall limit our detailed discussions to the ground states disposed along the parent trajectory. In this case, assuming that the potentials are sufficiently smooth and the bound states are well defined, the wavefunction does not contain any zeros, so its logarithm is regular and we can put

$$U(r) = \exp[S(r)/\hbar]. \tag{4}$$

It is then found by direct substitution that equation (2) takes the Riccati form

$$\hbar C'(r) + C^2(r) = G(r, E) + \Lambda/r^2 \tag{5}$$

where $C(r) = S'(r)$.

By analogy with the non-relativistic treatment, we seek a semiclassical solution to (5) by expanding the logarithmic derivative of the wavefunction in power series in Planck's constant $\hbar$:

$$C(r) = \sum_{k=0}^{\infty} C_k(r)\hbar^k. \tag{6}$$

So we are interested in Regge trajectories, and assume that $\Lambda = \hbar^2 l(l+1)$ is a function of the energy variable E and Planck's constant, and may also be written as an $\hbar$-expansion series

$$\Lambda = \Lambda(E, \hbar) = \sum_{k=0}^{\infty} \Lambda_k(E)\hbar^k. \tag{7}$$

Through the use of the $\hbar$-expansions (6) and (7), on comparing coefficients of various powers in $\hbar$, from equation (5) we then obtain

$$\begin{aligned} &C_0^2(r) = G(r, E) + \Lambda_0(E)/r^2 \\ &C_0'(r) + 2C_0(r)C_1(r) = \Lambda_1(E)/r^2 \\ &\dots \\ &C_{k-1}'(r) + \sum_{i=0}^{k} C_i(r)C_{k-i}(r) = \Lambda_k(E)/r^2. \end{aligned} \tag{8}$$

The recurrence system at hand is solvable if we know $\Lambda_0(E)$, which determines, as was treated earlier by Kastrup (1983), the bifurcation (or focal, or critical) curve in the angular momentum–energy plane.

Let our system be a classical mechanical system in a ground state with angular momentum L, and $L^2 = \Lambda_0$. This state, by definition, is the state with minimal energy. So, in the classical limit, $\hbar = 0$, the Klein–Gordon particle executes a circular motion with energy

$$E_0(L) = \sqrt{(m + V_s(r_0))^2 + L^2/r_0^2} + V_v(r_0) \tag{9}$$

where r_0 is the radius of the stable circular orbit.

For energies E above E_0 we will have a bounded motion, but for $E < E_0$ there is no motion at all. Consequently, the curve $E_0(L)$ defines a bifurcation set in the (L, E) plane and will play the part of the leading approximation for the Regge trajectories. It may be made to appear more explicitly in our designations as

$$\Lambda_0(E) = \frac{r_0^3}{2}\frac{\mathrm{d}}{\mathrm{d}r} G(r, E)|_{r=r_0} = \frac{r_0^3}{2} G'(r_0, E) \tag{10}$$

where $r_0 = r_0(E)$ is the real positive root of the equation

$$G(r_0, E) + \frac{\Lambda_0(E)}{r_0^2} = G(r_0, E) + \frac{r_0}{2} G'(r_0, E) = 0. \tag{11}$$

Then from the first equation of the system (8) in the neigbourhood of the point r_0 we have

$$C_0(r) = -\left(\frac{r-r_0}{r_0}\right)\left\{g_0(E) + \left(\frac{r-r_0}{r_0}\right) g_1(E) + \ldots\right\}^{1/2} \tag{12}$$

where the minus sign is chosen from a boundary condition, and

$$g_k(E) = \frac{r_0^{k+2}}{(k+2)!} G^{(k+2)}(r_0, E) + (-1)^k \frac{k+3}{2} r_0 G'(r_0), \qquad k = 0, 1, 2, \ldots \tag{13}$$

Due to $C_0(r_0) = 0$, the second equation from the system (8) gives immediately

$$\Lambda_1(E) = -r_0 g_0^{1/2}(E). \tag{14a}$$

Next we find $C_1(r)$ and pass to the third equation from (8), and so on, which results in

$$\Lambda_2(E) = \tfrac{5}{2} + \tfrac{3}{2}a_1 - \tfrac{3}{4}a_2 + \tfrac{11}{16}a_1^2 \tag{14b}$$

$$\begin{aligned}\Lambda_3(E) = -\frac{2}{\Lambda_1(E)}[&\tfrac{41}{16} + \tfrac{17}{8}a_1 - \tfrac{21}{16}a_2 + \tfrac{15}{8}a_3 - \tfrac{15}{16}a_4 + \tfrac{91}{64}a_1^2 + \tfrac{21}{32}a_2^2 - \tfrac{53}{16}a_1a_2 \\ &+ \tfrac{65}{32}a_1a_3 + \tfrac{99}{64}a_1^3 - \tfrac{171}{64}a_1^2a_2 + \tfrac{465}{512}a_1^4]\end{aligned} \tag{14c}$$

where $a_k = g_k(E)/g_0(E)$.

Finally the $\hbar$-expansion for the Regge trajectory takes the form

$$\alpha(E) = \alpha_0(E) + \hbar\alpha_1(E) + \hbar^2\alpha_2(E) + \ldots \tag{15a}$$

with

$$\alpha_0(E) = \Lambda_0^{1/2}(E) \tag{15b}$$

$$\alpha_1(E) = \frac{1}{2\alpha_0(E)}\{\Lambda_1(E) - \alpha_0(E)\} \tag{15c}$$

$$\ldots$$

$$\alpha_k(E) = \frac{1}{2\alpha_0(E)}\left\{\Lambda_k(E) - \alpha_{k-1}(E) - \sum_{i=1}^{k-1} \alpha_i(E)\alpha_{k-i}^{(E)}\right\} \qquad k = 1, 2, 3, \ldots. \tag{15d}$$

Notice that the expression for $\alpha(E)$ can be inverted at fixed l-values to yield the energy eigenvalues $E_{n,l}$. To illustrate the method, we consider some examples.

(1) π-mesonic atom. This classical example is interesting since it has an exact solution. There is only the vector potential $V_v(r) = -Ze^2/r$ in this case. Consequently we have

$$G(r, E) = m^2 - \left(E + \frac{Ze^2}{r}\right)^2 \tag{16}$$

and from equations (11) and (13) it follows that

$$r_0(E) = Ze^2 \frac{E}{m^2 - E^2} \qquad \Lambda_0(E) = (Ze^2)^2 \frac{m^2}{m^2 - E^2}$$

$$g_k(E) = (-1)^k (k+1)(m^2 - E^2) \qquad a_k = (-1)^k (k+1) \qquad k = 1, 2, \ldots .$$

Then the $\hbar$-expansion for $\Lambda(E, \hbar)$ takes the form

$$\Lambda(E, \hbar) = \hbar^2 l(l+1) = \frac{(Ze^2 m)^2}{m^2 - E^2} - \frac{Ze^2 E}{(m^2 - E^2)^{1/2}} \hbar. \tag{17}$$

After transforming this relation

$$\hbar^2 (l + \tfrac{1}{2})^2 = \left(\frac{Ze^2 E}{(m^2 - E^2)^{1/2}} - \frac{\hbar}{2}\right)^2 + (Ze^2)^2 \tag{18}$$

we receive the exact result for the relativistic ground state π-mesonic atom energies:

$$E_{0,l} = m\left(1 + \frac{Z^2 e^4}{\lambda^2}\right)^{-1/2} \tag{19}$$

where

$$\lambda = \frac{\hbar}{2} + (\hbar^2 (l + \tfrac{1}{2})^2 - (Ze^2)^2)^{1/2} \qquad l = 0, 1, 2, \ldots,$$

which is familiar from standard quantum mechanics textbooks.

(2) Funnel-shaped potential. Taking the increasing part of the potential as a Lorentz scalar, we write for the potential function

$$G(r, E) = (m + br)^2 - (E + q/r)^2. \tag{20}$$

Then it follows that

$$\Lambda_0(E) = b^2 r_0^4 + mbr_0^3 + qEr_0 + q^2 \tag{21}$$

and

$$\Lambda_1(E) = -(4b^2 r_0^4 + 3mbr_0^3 + qEr_0)^{1/2} \tag{22}$$

where r_0 is the real positive root of the equation

$$2b^2 r_0^3 + 3bmr_0^2 + (m^2 - E^2) r_0 - qE = 0. \tag{23}$$

The following terms of the expansion for Λ are derived from equations (14*b*) and (14*c*) by substitution

$$a_1 = -2\delta(1 + 2\beta + 2\gamma) \tag{24a}$$

$$a_2 = \delta(3 + 5\beta + 5\gamma) \tag{24b}$$

$$a_3 = -2\delta(2 + 3\beta + 3\gamma) \tag{24c}$$

$$a_4 = \delta(5 + 7\beta + 7\gamma) \tag{24d}$$

Table 1. The values of the parent Regge trajectory with one, two and three quantum corrections ($\alpha^{(1)} = \alpha_0 + \hbar\alpha_1$, $\alpha^{(2)} = \alpha^{(1)} + \hbar^2\alpha_2$, $\alpha^{(3)} = \alpha^{(2)} + \hbar^3\alpha_3$) computed for the funnel-shaped potential at the energies $E_{0,l}^{\mathrm{num}}$, obtained by numerical integration (Krasemann 1981). The eigenvalues $E_{0,l}^{(1)}$, $E_{0,l}^{(2)}$, $E_{0,l}^{(3)}$ are computed by inverting formulae for $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$ with fixed l-values.

l	$\alpha^{(1)}$	$\alpha^{(2)}$	$\alpha^{(3)}$	$E_{0,l}^{(1)}$	$E_{0,l}^{(2)}$	$E_{0,l}^{(3)}$	$E_{0,l}^{\mathrm{num}}$
0	−0.0493	−0.0153	0.0060	1.5494	1.5389	1.5356	1.533
1	0.9900	0.9985	1.0006	1.7621	1.7607	1.7604	1.760
2	1.9976	2.0008	2.0015	1.9043	1.9039	1.9038	1.904

where

$$\beta = (bm/qE)r_0^2 \qquad \gamma = (b^2/qE)r_0^3 \qquad \delta = 1 + 3\beta + 4\gamma.$$

Table 1 illustrates the speed and accuracy of our technique on the parent Regge trajectory calculation with one, two and three quantum corrections, computed for the funnel-shaped potential with parameters $m = 1.370$ GeV, $b = 0.104\,29$ (GeV)^2, $q = 0.26$ at the energies $E_{0,l}^{\mathrm{num}}$ (in GeV), obtained by numerical integration (Krasemann 1981). The eigenvalues $E_{0,l}^{(1)}$, $E_{0,l}^{(2)}$, $E_{0,l}^{(3)}$, computed at fixed l-values by the formulae for the Regge trajectory with one, two and three quantum corrections, are displayed as well. It is seen that listed values are in good agreement with exact ones, manifesting the improvement with increasing l.

In conclusion, we have extended the $\hbar$-expansion for parent Regge trajectories, developed recently within non-relativistic quantum mechanics, to the relativistic case in which the Klein–Gordon equation is used. For the Coulomb potential this technique is found to give the exact result. Numerical work with the funnel-shaped potential proves that first terms of the expansions already provide high accuracy for the spectrum. The generalisation to an arbitrary excited bound state is to be published elsewhere.

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